Natural frequencies of double section beam, with mass at the top

$M_{trolley} := 3300\ \mathrm{kg}$	suspended mass
$l := 0.01\ \mathrm{m}$	position of suspended mass from the top
$A_1 := 192\ \mathrm{cm}^2$	bottom column cross section area
$I_1 := 65810\ \mathrm{cm}^4$	bottom column cross section inertia
$l_1 := 7.4\ \mathrm{m}$	bottom column lenght
$A_2 := 192\ \mathrm{cm}^2$	top column cross section area
$I_2 := 65810\ \mathrm{cm}^4$	top column cross section inertia
$l_2 := 7.4\ \mathrm{m}$	top column lenght
$E := 206000\ \dfrac{\mathrm{N}}{\mathrm{mm}^2}$	Hooke modulus
$\rho := 7850\ \dfrac{\mathrm{kg}}{\mathrm{m}^3}$	density

$$\alpha := \frac{l_2 - l}{l_2} = 0.998648648648649$$

ELEMENTS STIFFNESS MATRIX

$$K_{11} := \frac{\dfrac{E}{\frac{\mathrm{N}}{\mathrm{m}^2}} \cdot \dfrac{I_1}{\mathrm{m}^4}}{\left(\dfrac{\frac{l_1}{2}}{\mathrm{m}}\right)^3} \cdot \begin{bmatrix} 12 & 6 \cdot \dfrac{\frac{l_1}{2}}{\mathrm{m}} & -12 & 6 \cdot \dfrac{\frac{l_1}{2}}{\mathrm{m}} \\ 6 \cdot \dfrac{\frac{l_1}{2}}{\mathrm{m}} & 4 \cdot \left(\dfrac{\frac{l_1}{2}}{\mathrm{m}}\right)^2 & -6 \cdot \dfrac{\frac{l_1}{2}}{\mathrm{m}} & 2 \cdot \left(\dfrac{\frac{l_1}{2}}{\mathrm{m}}\right)^2 \\ -12 & -6 \cdot \dfrac{\frac{l_1}{2}}{\mathrm{m}} & 12 & -6 \cdot \dfrac{\frac{l_1}{2}}{\mathrm{m}} \\ 6 \cdot \dfrac{\frac{l_1}{2}}{\mathrm{m}} & 2 \cdot \left(\dfrac{\frac{l_1}{2}}{\mathrm{m}}\right)^2 & -6 \cdot \dfrac{\frac{l_1}{2}}{\mathrm{m}} & 4 \cdot \left(\dfrac{\frac{l_1}{2}}{\mathrm{m}}\right)^2 \end{bmatrix}$$

$$K_{12} := \frac{\dfrac{E}{\frac{\mathrm{N}}{\mathrm{m}^2}} \cdot \dfrac{I_1}{\mathrm{m}^4}}{\left(\dfrac{\frac{l_1}{2}}{\mathrm{m}}\right)^3} \cdot \begin{bmatrix} 12 & 6 \cdot \dfrac{\frac{l_1}{2}}{\mathrm{m}} & -12 & 6 \cdot \dfrac{\frac{l_1}{2}}{\mathrm{m}} \\ 6 \cdot \dfrac{\frac{l_1}{2}}{\mathrm{m}} & 4 \cdot \left(\dfrac{\frac{l_1}{2}}{\mathrm{m}}\right)^2 & -6 \cdot \dfrac{\frac{l_1}{2}}{\mathrm{m}} & 2 \cdot \left(\dfrac{\frac{l_1}{2}}{\mathrm{m}}\right)^2 \\ -12 & -6 \cdot \dfrac{\frac{l_1}{2}}{\mathrm{m}} & 12 & -6 \cdot \dfrac{\frac{l_1}{2}}{\mathrm{m}} \end{bmatrix}$$

$$\begin{bmatrix} 6\cdot\dfrac{\frac{l_1}{2}}{\mathrm{m}} & 2\cdot\left(\dfrac{\frac{l_1}{2}}{\mathrm{m}}\right) & -6\cdot\dfrac{\frac{l_1}{2}}{\mathrm{m}} & 4\cdot\left(\dfrac{\frac{l_1}{2}}{\mathrm{m}}\right) \end{bmatrix}$$

$$K_{21} := \frac{\dfrac{E}{\frac{\mathrm{N}}{\mathrm{m}^2}}\cdot\dfrac{I_2}{\mathrm{m}^4}}{\left(\dfrac{\alpha\cdot l_2}{\mathrm{m}}\right)^3}\cdot\begin{bmatrix} 12 & 6\cdot\dfrac{\alpha\cdot l_2}{\mathrm{m}} & -12 & 6\cdot\dfrac{\alpha\cdot l_2}{\mathrm{m}} \\ 6\cdot\dfrac{\alpha\cdot l_2}{\mathrm{m}} & 4\cdot\left(\dfrac{\alpha\cdot l_2}{\mathrm{m}}\right)^2 & -6\cdot\dfrac{\alpha\cdot l_2}{\mathrm{m}} & 2\cdot\left(\dfrac{\alpha\cdot l_2}{\mathrm{m}}\right)^2 \\ -12 & -6\cdot\dfrac{\alpha\cdot l_2}{\mathrm{m}} & 12 & -6\cdot\dfrac{\alpha\cdot l_2}{\mathrm{m}} \\ 6\cdot\dfrac{\alpha\cdot l_2}{\mathrm{m}} & 2\cdot\left(\dfrac{\alpha\cdot l_2}{\mathrm{m}}\right)^2 & -6\cdot\dfrac{\alpha\cdot l_2}{\mathrm{m}} & 4\cdot\left(\dfrac{\alpha\cdot l_2}{\mathrm{m}}\right)^2 \end{bmatrix}$$

$$K_{22} := \frac{\dfrac{E}{\frac{\mathrm{N}}{\mathrm{m}^2}}\cdot\dfrac{I_2}{\mathrm{m}^4}}{\left(\dfrac{(1-\alpha)\cdot l_2}{\mathrm{m}}\right)^3}\cdot\begin{bmatrix} 12 & 6\cdot\dfrac{(1-\alpha)\cdot l_2}{\mathrm{m}} & -12 & 6\cdot\dfrac{(1-\alpha)\cdot l_2}{\mathrm{m}} \\ 6\cdot\dfrac{(1-\alpha)\cdot l_2}{\mathrm{m}} & 4\cdot\left(\dfrac{(1-\alpha)\cdot l_2}{\mathrm{m}}\right)^2 & -6\cdot\dfrac{(1-\alpha)\cdot l_2}{\mathrm{m}} & 2\cdot\left(\dfrac{(1-\alpha)\cdot l_2}{\mathrm{m}}\right)^2 \\ -12 & -6\cdot\dfrac{(1-\alpha)\cdot l_2}{\mathrm{m}} & 12 & -6\cdot\dfrac{(1-\alpha)\cdot l_2}{\mathrm{m}} \\ 6\cdot\dfrac{(1-\alpha)\cdot l_2}{\mathrm{m}} & 2\cdot\left(\dfrac{(1-\alpha)\cdot l_2}{\mathrm{m}}\right)^2 & -6\cdot\dfrac{(1-\alpha)\cdot l_2}{\mathrm{m}} & 4\cdot\left(\dfrac{(1-\alpha)\cdot l_2}{\mathrm{m}}\right)^2 \end{bmatrix}$$

SYSTEM STIFFNESS MATRIX

```
merge(base, data, row, col) := | out := base
                               | for  r ∈ [1..rows(data)]
                               |   for  c ∈ [1..cols(data)]
                               |     out                      := data
                               |        row+r-1  col+c-1          r c
                               | out
```

function to assembly matrix

$$K := \mathrm{matrix}(10, 10) = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_{system} := merge\left(K, K_{11\,[1..4][1..4]}, 1, 1\right) + merge\left(K, K_{12\,[1..4][1..4]}, 3, 3\right) + merge\left(K, K_{21\,[1..4][1..}\right.$$

CONSTARINED SYSTEM STIFFNESS MATRIX

$$m := \text{rows}\left(K_{system}\right) = 10$$

$$K_{system\_cons} := \text{submatrix}\left(K_{system}, 3, m, 3, m\right)$$

ELEMENTS MASS MATRIX

$$M_{11} := \frac{\frac{\rho}{\frac{\text{kg}}{\text{m}^3}} \cdot \frac{A_1}{\text{m}^2} \cdot \frac{l_1}{\frac{2}{\text{m}}}}{420} \cdot \begin{bmatrix} 156 & 22 \cdot \frac{l_1}{\frac{2}{\text{m}}} & 54 & -13 \cdot \frac{l_1}{\frac{2}{\text{m}}} \\ 22 \cdot \frac{l_1}{\frac{2}{\text{m}}} & 4 \cdot \left(\frac{l_1}{\frac{2}{\text{m}}}\right)^2 & 13 \cdot \frac{l_1}{\frac{2}{\text{m}}} & -3 \cdot \left(\frac{l_1}{\frac{2}{\text{m}}}\right)^2 \\ 54 & 13 \cdot \frac{l_1}{\frac{2}{\text{m}}} & 156 & -22 \cdot \frac{l_1}{\frac{2}{\text{m}}} \\ -13 \cdot \frac{l_1}{\frac{2}{\text{m}}} & -3 \cdot \left(\frac{l_1}{\frac{2}{\text{m}}}\right)^2 & -22 \cdot \frac{l_1}{\frac{2}{\text{m}}} & 4 \cdot \left(\frac{l_1}{\frac{2}{\text{m}}}\right) \end{bmatrix}$$

$$M_{12} := \frac{\frac{\rho}{\frac{\text{kg}}{\text{m}^3}} \cdot \frac{A_1}{\text{m}^2} \cdot \frac{l_1}{\frac{2}{\text{m}}}}{420} \cdot \begin{bmatrix} 156 & 22 \cdot \frac{l_1}{\frac{2}{\text{m}}} & 54 & -13 \cdot \frac{l_1}{\frac{2}{\text{m}}} \\ 22 \cdot \frac{l_1}{\frac{2}{\text{m}}} & 4 \cdot \left(\frac{l_1}{\frac{2}{\text{m}}}\right)^2 & 13 \cdot \frac{l_1}{\frac{2}{\text{m}}} & -3 \cdot \left(\frac{l_1}{\frac{2}{\text{m}}}\right)^2 \\ 54 & 13 \cdot \frac{l_1}{\frac{2}{\text{m}}} & 156 & -22 \cdot \frac{l_1}{\frac{2}{\text{m}}} \\ -13 \cdot \frac{l_1}{\frac{2}{\text{m}}} & -3 \cdot \left(\frac{l_1}{\frac{2}{\text{m}}}\right)^2 & -22 \cdot \frac{l_1}{\frac{2}{\text{m}}} & 4 \cdot \left(\frac{l_1}{\frac{2}{\text{m}}}\right) \end{bmatrix}$$

$$\frac{\rho}{\frac{\text{kg}}{\text{m}^3}} \cdot \frac{A_2}{\text{m}^2} \cdot \frac{\alpha \cdot l_2}{\text{m}} \quad \begin{bmatrix} 156 & 22 \cdot \frac{\alpha \cdot l_2}{\text{m}} & 54 & -13 \cdot \frac{\alpha \cdot l_2}{\text{m}} \\ 22 \cdot \frac{\alpha \cdot l_2}{} & 4 \cdot \left(\frac{\alpha \cdot l_2}{}\right)^2 & 13 \cdot \frac{\alpha \cdot l_2}{} & -3 \cdot \left(\frac{\alpha \cdot l_2}{}\right)^2 \end{bmatrix}$$

$$M_{21} := \frac{\ \mathrm{m}^{3}}{420} \cdot \begin{bmatrix} \ \mathrm{m} & 4 \cdot \left(\frac{}{\mathrm{m}} \right) & \mathrm{m} & -3 \cdot \left(\frac{}{\mathrm{m}} \right) \\ 54 & 13 \cdot \frac{\alpha \cdot l_2}{\mathrm{m}} & 156 & -22 \cdot \frac{\alpha \cdot l_2}{\mathrm{m}} \\ -13 \cdot \frac{\alpha \cdot l_2}{\mathrm{m}} & -3 \cdot \left(\frac{\alpha \cdot l_2}{\mathrm{m}} \right)^2 & -22 \cdot \frac{\alpha \cdot l_2}{\mathrm{m}} & 4 \cdot \left(\frac{\alpha \cdot l_2}{\mathrm{m}} \right) \end{bmatrix}$$

$$M_{22} := \frac{\frac{\rho}{\frac{\mathrm{kg}}{\mathrm{m}^3}} \cdot \frac{A_2}{\mathrm{m}^2} \cdot \frac{(1-\alpha) \cdot l_2}{\mathrm{m}}}{420} \cdot \begin{bmatrix} 156 & 22 \cdot \frac{(1-\alpha) \cdot l_2}{\mathrm{m}} & 54 & -13 \cdot \frac{(1-\alpha) \cdot l_2}{\mathrm{m}} \\ 22 \cdot \frac{(1-\alpha) \cdot l_2}{\mathrm{m}} & 4 \cdot \left(\frac{(1-\alpha) \cdot l_2}{\mathrm{m}} \right)^2 & 13 \cdot \frac{(1-\alpha) \cdot l_2}{\mathrm{m}} & -3 \cdot \left(\frac{(1-\alpha) \cdot l_2}{\mathrm{m}} \right)^2 \\ 54 & 13 \cdot \frac{(1-\alpha) \cdot l_2}{\mathrm{m}} & 156 & -22 \cdot \frac{(1-\alpha) \cdot l_2}{\mathrm{m}} \\ -13 \cdot \frac{(1-\alpha) \cdot l_2}{\mathrm{m}} & -3 \cdot \left(\frac{(1-\alpha) \cdot l_2}{\mathrm{m}} \right)^2 & -22 \cdot \frac{(1-\alpha) \cdot l_2}{\mathrm{m}} & 4 \cdot \left(\frac{(1-\alpha) \cdot l_2}{\mathrm{m}} \right) \end{bmatrix}$$

SYSTEM MASS MATRIX

```
merge(base, data, row, col) := | out := base
                               | for  r ∈ [1..rows(data)]
                               |   for  c ∈ [1..cols(data)]
                               |     out[row+r-1, col+c-1] := data[r, c]
                               | out
```

function to assembly matrix

$$M := \mathrm{matrix}(10, 10) = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$M_{7\,7} := \frac{M_{trolley}}{\mathrm{kg}}$$

$$M_{system} := merge\left(M, {M_{11}}_{[1..4][1..4]}, 1, 1\right) + merge\left(M, {M_{12}}_{[1..4][1..4]}, 3, 3\right) + merge\left(M, {M_{21}}_{[1..4][1..$$

CONSTARINED SYSTEM MASS MATRIX

$$q := \mathrm{rows}(M\ \ ) = 10$$

$(\; system\;)$

$M_{system\_cons} := \mathrm{submatrix}\left(M_{system}, 3, q, 3, q\right)$

DYNAMICAL MATRIX

$A := \mathrm{invert}\left(K_{system\_cons}\right) \cdot M_{system\_cons}$

EIGENVALUE EVALUATION

```
MEigen(A) := | [ n := rows(A)  C := A  Id := identity(n)  v := matrix(n, 1)  s := v  v₁ := tr(A) ]
             | [ E := matrix(n, n)  R := matrix(n − 1, n − 1)  ro := [1..n] ]
             | for  k := 2, k ≤ n, k := k + 1
             |    [ C := A·(C − v_(k−1)·Id)   v_k := tr(C)/k ]
             | λ := polyroots(stack(−reverse(v), 1))
             | for  k ∈ ro
             |    | kOut(X) := | [ j := 0  W := matrix(n − 1, 1) ]      kIn(X) := | [ j := 0  W := matrix(n, 1) ]
             |    |            | for  r ∈ ro                                     | for  r ∈ ro
             |    |            |   if  r ≠ k                                     |   if  r ≠ k
             |    |            |     W_j := X_r   j := j + 1                     |     W_r := X_j   j := j + 1
             |    |            |   else                                          |   else
             |    |            |     continue                                    |     W_r := 1
             |    |            | W                                               | W / norme(W)
             |    | v_ro := str2num(concat("_MEIGENV#_", num2str(ro)))
             |    | [ u := col(Id, k)  v_k := 1  eq := kOut(A·v − λ_k·v)  vo := kOut(v) ]
             |    | [ r := [1..(n − 1)]  c := r ]
             |    | R_(r c) := str2num(concat("diff(", num2str(eq_r), ",", num2str(vo_c), ")"))
             |    | [ ε := try                                          | E_(ro k) := ε_ro
             |    |          kIn(−R⁻¹·kOut(A·u − λ_k·u))                 | s_k := norme(A·ε − λ_k·ε)
             |    |        on error
             |    |          matrix(n, 1) ]
             | [ λ  E ]
```

$$[\lambda\;\; M] := MEigen(\mathrm{eval}(A)) = \left[\begin{bmatrix} -9.6786\cdot 10^{-9} \\ 9.6786\cdot 10^{-9} \\ 1.3878\cdot 10^{-17} + 9.0295\cdot 10^{-9}\cdot i \\ 1.3878\cdot 10^{-17} - 9.0295\cdot 10^{-9}\cdot i \\ -4.4286\cdot 10^{-9} \\ 1.6411\cdot 10^{-5} \\ 0.0002 \\ 0.0567 \end{bmatrix} \begin{bmatrix} 0.4059 & 0.0663 & -0.1511 + 0.0019\cdot i & 0.1422 + 0.0041\cdot i \\ 0.4426 & 0.9188 & -0.8666 - 0.0076\cdot i & 0.6626 + 0.0018\cdot i \\ -0.4743 & -0.2625 & 0.334 & -0.3536 - 0.0082\cdot i \\ 0.2251 & 0.1321 & -0.2004 + 0.001\cdot i & 0.1146 \\ -0.0247 & -0.0105 & 0.0118 + 7.3823\cdot 10^{-5}\cdot i & -0.0254 - 0.001\cdot i \\ -0.4256 & -0.18 & 0.1924 + 0.0015\cdot i & -0.4474 - 0.0173\cdot i \\ -0.0289 & -0.0123 & 0.0137 + 8.9184\cdot 10^{-5}\cdot i & -0.0299 - 0.0011\cdot i \\ -0.4256 & -0.18 & 0.1924 + 0.0015\cdot i & -0.4474 - 0.0173\cdot i \end{bmatrix}\right]$$

$$\text{norme}\left(A \cdot M - M \cdot \text{diag}(\lambda)\right) = 0$$

or better

$$[\,\lambda\ M\,] := \overrightarrow{\text{Re}\left(MEigen\left(\text{eval}(A)\right)\right)} = \left[\begin{bmatrix} -9.6786 \cdot 10^{-9} \\ 9.6786 \cdot 10^{-9} \\ 1.3878 \cdot 10^{-17} \\ 1.3878 \cdot 10^{-17} \\ -4.4286 \cdot 10^{-9} \\ 1.6411 \cdot 10^{-5} \\ 0.0002 \\ 0.0567 \end{bmatrix} \begin{bmatrix} 0.4059 & 0.0663 & -0.1511 & 0.1422 & -3.9864 \cdot 10^{-6} & 0.7331 & 0.1166 & 0.059 \\ 0.4426 & 0.9188 & -0.8666 & 0.6626 & -1.6639 \cdot 10^{-5} & 0.1287 & 0.0476 & 0.030 \\ -0.4743 & -0.2625 & 0.334 & -0.3536 & 5.7359 \cdot 10^{-5} & 0.1255 & 0.2535 & 0.215 \\ 0.2251 & 0.1321 & -0.2004 & 0.1146 & 3.855 \cdot 10^{-5} & -0.3792 & 0.0164 & 0.052 \\ -0.0247 & -0.0105 & 0.0118 & -0.0254 & 0.0001 & 0.0404 & -0.0196 & 0.684 \\ -0.4256 & -0.18 & 0.1924 & -0.4474 & 0.0003 & 0.4414 & -0.0682 & 0.068 \\ -0.0289 & -0.0123 & 0.0137 & -0.0299 & -1 & 0.0434 & 0.9539 & 0.684 \\ -0.4256 & -0.18 & 0.1924 & -0.4474 & 0.0004 & 0.2969 & -0.0682 & 0.068 \end{bmatrix}\right.$$

$$\text{norme}\left(A \cdot M - M \cdot \text{diag}(\lambda)\right) = 0$$

$$\lambda := \text{dn\_LinAlgEigenvalues}(A) = \begin{bmatrix} 0.0567 \\ 0.0002 \\ 1.5588 \cdot 10^{-5} \\ 2.1646 \cdot 10^{-6} \\ -1.5921 \cdot 10^{-6} \\ 1.9853 \cdot 10^{-7} \\ 1.1 \cdot 10^{-14} \\ 3.2702 \cdot 10^{-16} \end{bmatrix}$$

$$M := \text{dn\_LinAlgEigenvectors}(A) = \begin{bmatrix} 0.0594 & -0.3876 & 0.7642 & 0.4958 & 0.019 & 0.0813 & -1.7029 \cdot 10^{-7} & -1.7636 \cdot 10^{-6} \\ 0.0306 & -0.1583 & 0.1268 & -0.3112 & 0.0173 & 0.908 & -6.5593 \cdot 10^{-7} & -6.7948 \cdot 10^{-6} \\ 0.2154 & -0.8428 & 0.0845 & -0.4867 & -0.1408 & -0.2773 & 1.3133 \cdot 10^{-7} & 1.4177 \cdot 10^{-6} \\ 0.0522 & -0.0544 & -0.4042 & 0.3519 & -0.2257 & 0.1454 & -9.5712 \cdot 10^{-7} & -9.848 \cdot 10^{-6} \\ 0.6841 & 0.0649 & 0.0354 & -0.0257 & -0.0339 & -0.0111 & 3.4654 \cdot 10^{-7} & 7.2782 \cdot 10^{-6} \\ 0.0689 & 0.2267 & 0.3366 & -0.3841 & -0.6804 & -0.188 & -9.1124 \cdot 10^{-7} & -7.1421 \cdot 10^{-6} \\ 0.6848 & 0.0672 & 0.0388 & -0.0296 & -0.0407 & -0.013 & -0.0051 & -0.0671 \\ 0.0689 & 0.2267 & 0.3366 & -0.3841 & -0.6804 & -0.188 & -1 & 0.9977 \end{bmatrix}$$

$$\text{norme}\left(A \cdot M - M \cdot \text{diag}(\lambda)\right) = 7 \cdot 10^{-16}$$

NATURAL FREQUENCIES

$$n := \text{rows}(A)$$

$$\text{for } j \in [1..n]$$

$$f_j := \frac{\sqrt{\dfrac{1}{\text{dn\_LinAlgEigenvalues}(A)_j}}}{2 \cdot \pi}$$

$$\begin{bmatrix} 0.6686 \\ 11.1367 \\ 40.3105 \\ 108.1762 \end{bmatrix}$$

$$f = \begin{bmatrix} -126.1343 \cdot i \\ 357.1928 \\ 1.5175 \cdot 10^{6} \\ 8.8011 \cdot 10^{6} \end{bmatrix}$$

$$f_1 := f_1 \ \mathrm{Hz}$$

$$f_2 := f_2 \ \mathrm{Hz}$$

$$f_3 := f_3 \ \mathrm{Hz}$$

$$f_4 := f_4 \ \mathrm{Hz}$$

CHECKING VALUE: EXACT VALUE OF NATURAL FREQUENCIES (UNIFORM CROSS SECTION BEAM and WITHOUT SUSPENDED MASS OF THE TROLLEY

$$f_{1exact} := \frac{(1.875)^2}{2 \cdot \pi} \cdot \sqrt{\frac{E \cdot I_1}{\rho \cdot A_1 \cdot \left(l_1 + l_2\right)^4}} = 2.4227 \ \mathrm{Hz} \qquad \frac{f_1 - f_{1exact}}{f_1} = -262.3251 \ \%$$

$$f_{2exact} := \frac{(4.694)^2}{2 \cdot \pi} \cdot \sqrt{\frac{E \cdot I_1}{\rho \cdot A_1 \cdot \left(l_1 + l_2\right)^4}} = 15.1837 \ \mathrm{Hz} \qquad \frac{f_2 - f_{2exact}}{f_2} = -36.3388 \ \%$$

$$f_{3exact} := \frac{(7.855)^2}{2 \cdot \pi} \cdot \sqrt{\frac{E \cdot I_1}{\rho \cdot A_1 \cdot \left(l_1 + l_2\right)^4}} = 42.519 \ \mathrm{Hz} \qquad \frac{f_3 - f_{3exact}}{f_3} = -5.4788 \ \%$$

MODE SHAPES

$$w_{11\_1}(x) := \left(3 \cdot \left(\frac{x}{\frac{l_1}{2}\mathrm{m}}\right)^2 - 2 \cdot \left(\frac{x}{\frac{l_1}{2}\mathrm{m}}\right)^3\right) \cdot \mathrm{dn\_LinAlgEigenvectors}(A)_{1\,1} + \left(-\left(\frac{x}{\frac{l_1}{2}\mathrm{m}}\right)^2 + \left(\frac{x}{\frac{l_1}{2}\mathrm{m}}\right)^3\right) \cdot \frac{l_1}{2}\mathrm{m} \cdot \mathrm{dn\_LinAlgEi}$$

$$w_{element1\_1}(x) := w_{11\_1}(x)$$

$$w_{12\_1}(x) := \left(1 - 3 \cdot \left(\frac{x}{\frac{l_1}{2}\mathrm{m}}\right)^2 + 2 \cdot \left(\frac{x}{\frac{l_1}{2}\mathrm{m}}\right)^3\right) \cdot \mathrm{dn\_LinAlgEigenvectors}(A)_{1\,1} + \left(\frac{x}{\frac{l_1}{2}\mathrm{m}} - 2 \cdot \left(\frac{x}{\frac{l_1}{2}\mathrm{m}}\right)^2 + \left(\frac{x}{\frac{l_1}{2}\mathrm{m}}\right)^3\right) \cdot \frac{l_1}{2}\mathrm{m} \cdot \mathrm{dn\_}$$

$$w_{element2\_1}(x) := w_{12\_1}\left(x - \frac{\frac{l_1}{2}}{\mathrm{m}}\right)$$

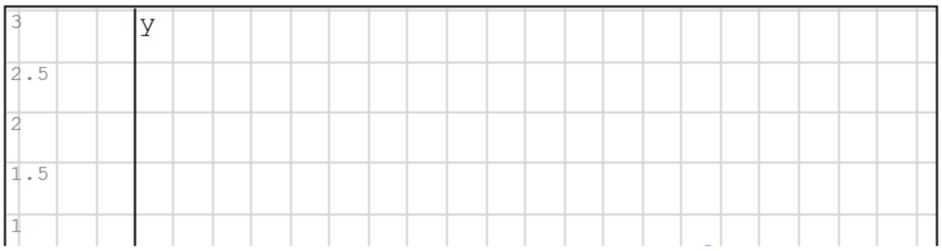
$$w_{21\_1}(x) := \left(1 - 3\cdot\left(\frac{x}{\frac{\alpha\cdot l_2}{\mathrm{m}}}\right)^2 + 2\cdot\left(\frac{x}{\frac{\alpha\cdot l_2}{\mathrm{m}}}\right)^3\right)\cdot \mathrm{dn\_LinAlgEigenvectors}(A)_{3\,1} + \left(\frac{x}{\frac{\alpha\cdot l_2}{\mathrm{m}}} - 2\cdot\left(\frac{x}{\frac{\alpha\cdot l_2}{\mathrm{m}}}\right)^2 + \left(\frac{x}{\frac{\alpha\cdot l_2}{\mathrm{m}}}\right.\right.$$

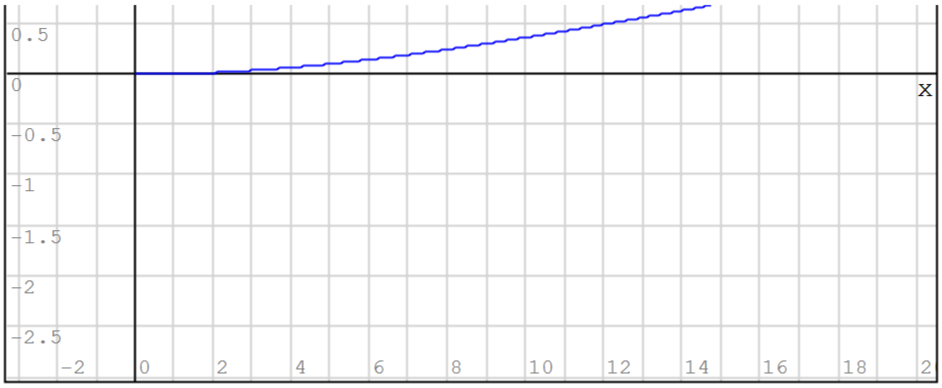
$$w_{element3\_1}(x) := w_{21\_1}\left(x - \frac{l_1}{\mathrm{m}}\right)$$

$$w_{22\_1}(x) := \left(1 - 3\cdot\left(\frac{x}{\frac{(1-\alpha)\cdot l_2}{\mathrm{m}}}\right)^2 + 2\cdot\left(\frac{x}{\frac{(1-\alpha)\cdot l_2}{\mathrm{m}}}\right)^3\right)\cdot \mathrm{dn\_LinAlgEigenvectors}(A)_{5\,1} + \left(\frac{x}{\frac{(1-\alpha)\cdot l_2}{\mathrm{m}}} - 2\cdot\left(\frac{x}{\frac{(1-\alpha}{\mathrm{m}}}\right.\right.$$

$$w_{element4\_1}(x) := w_{22\_1}\left(x - \frac{l_1 + \alpha\cdot l_2}{\mathrm{m}}\right)$$

$$w_{first}(x) := \begin{array}{l} \text{if } (x \geq 0) \wedge \left(x < \frac{\frac{l_1}{2}}{\mathrm{m}}\right) \\ \quad w_{element1\_1}(x) \\ \text{else} \\ \quad \text{if } \left(x \geq \frac{\frac{l_1}{2}}{\mathrm{m}}\right) \wedge \left(x \leq \frac{l_1}{\mathrm{m}}\right) \\ \quad\quad w_{element2\_1}(x) \\ \quad \text{else} \\ \quad\quad \text{if } \left(x > \frac{l_1}{\mathrm{m}}\right) \wedge \left(x \leq \frac{l_1 + \alpha\cdot l_2}{\mathrm{m}}\right) \\ \quad\quad\quad w_{element3\_1}(x) \\ \quad\quad \text{else} \\ \quad\quad\quad \text{if } \left(x > \frac{l_1 + \alpha\cdot l_2}{\mathrm{m}}\right) \wedge \left(x \leq \frac{l_1 + l_2}{\mathrm{m}}\right) \\ \quad\quad\quad\quad w_{element3\_1}(x) \\ \quad\quad\quad \text{else} \\ \quad\quad\quad\quad \text{"not defined"} \end{array}$$





$w_{first}(x)$

$$w_{11\_2}(x) := \left(3\cdot\left(\frac{x}{\frac{\frac{l_1}{2}}{m}}\right)^2 - 2\cdot\left(\frac{x}{\frac{\frac{l_1}{2}}{m}}\right)^3\right)\cdot \mathrm{dn\_LinAlgEigenvectors}(A)_{1\,2} + \left(-\left(\frac{x}{\frac{\frac{l_1}{2}}{m}}\right)^2 + \left(\frac{x}{\frac{\frac{l_1}{2}}{m}}\right)^3\right)\cdot\frac{\frac{l_1}{2}}{m}\cdot \mathrm{dn\_LinAl}$$

$$w_{element1\_2}(x) := w_{11\_2}(x)$$

$$w_{12\_2}(x) := \left(1 - 3\cdot\left(\frac{x}{\frac{\frac{l_1}{2}}{m}}\right)^2 + 2\cdot\left(\frac{x}{\frac{\frac{l_1}{2}}{m}}\right)^3\right)\cdot \mathrm{dn\_LinAlgEigenvectors}(A)_{1\,2} + \left(\frac{x}{\frac{\frac{l_1}{2}}{m}} - 2\cdot\left(\frac{x}{\frac{\frac{l_1}{2}}{m}}\right)^2 + \left(\frac{x}{\frac{\frac{l_1}{2}}{m}}\right)^3\right)\cdot\frac{\frac{l_1}{2}}{m}$$

$$w_{element2\_2}(x) := w_{12\_2}\left(x - \frac{\frac{l_1}{2}}{m}\right)$$

$$w_{21\_2}(x) := \left(1 - 3\cdot\left(\frac{x}{\frac{\alpha\cdot l_2}{m}}\right)^2 + 2\cdot\left(\frac{x}{\frac{\alpha\cdot l_2}{m}}\right)^3\right)\cdot \mathrm{dn\_LinAlgEigenvectors}(A)_{3\,2} + \left(\frac{x}{\frac{\alpha\cdot l_2}{m}} - 2\cdot\left(\frac{x}{\frac{\alpha\cdot l_2}{m}}\right)^2 + \left(\frac{x}{\frac{\alpha\cdot \ldots}{m}}\right.\right.$$

$$w_{element3\_2}(x) := w_{21\_2}\left(x - \frac{l_1}{m}\right)$$

$$w_{22\_2}(x) := \left(1 - 3\cdot\left(\frac{x}{\frac{(1-\alpha)\cdot l_2}{m}}\right)^2 + 2\cdot\left(\frac{x}{\frac{(1-\alpha)\cdot l_2}{m}}\right)^3\right)\cdot \mathrm{dn\_LinAlgEigenvectors}(A)_{5\,2} + \left(\frac{x}{\frac{(1-\alpha)\cdot l_2}{m}} - 2\cdot\left(\ldots\right.\right.$$

$$\left(\quad l_1 + \alpha\cdot l_2\right)$$

$$w_{element4\_2}(x) := w_{22\_2}\left(x - \frac{l_1 \cdot \quad l_2}{m}\right)$$

$$w_{second}(x) := \text{if } (x \geq 0) \wedge \left(x < \frac{\frac{l_1}{2}}{m}\right)$$

$$\quad w_{element1\_2}(x)$$

$$\quad \text{else}$$

$$\quad\quad \text{if } \left(x \geq \frac{\frac{l_1}{2}}{m}\right) \wedge \left(x \leq \frac{l_1}{m}\right)$$

$$\quad\quad\quad w_{element2\_2}(x)$$

$$\quad\quad \text{else}$$

$$\quad\quad\quad \text{if } \left(x > \frac{l_1}{m}\right) \wedge \left(x \leq \frac{l_1 + \alpha \cdot l_2}{m}\right)$$

$$\quad\quad\quad\quad w_{element3\_2}(x)$$

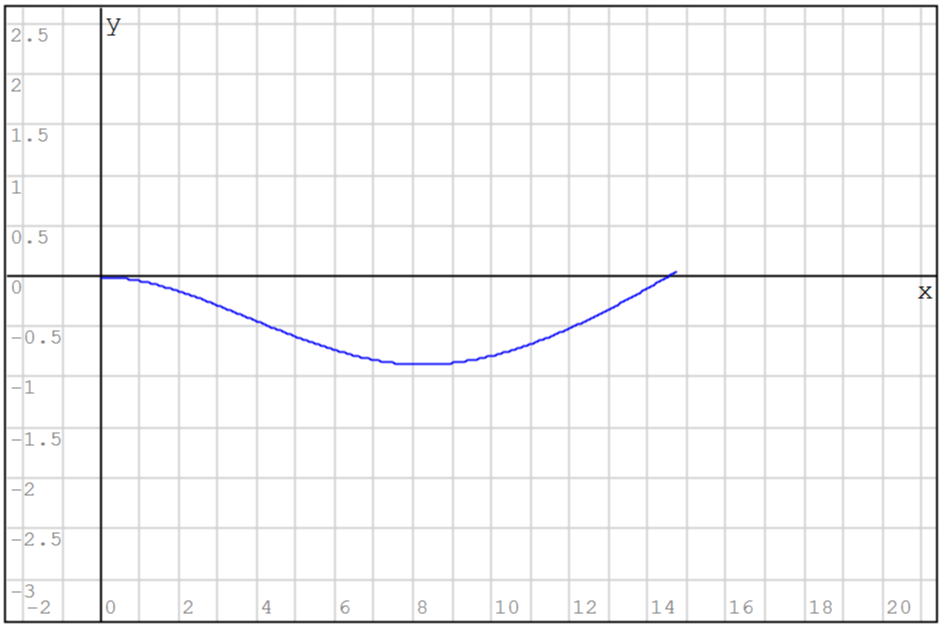
$$\quad\quad\quad \text{else}$$

$$\quad\quad\quad\quad \text{if } \left(x > \frac{l_1 + \alpha \cdot l_2}{m}\right) \wedge \left(x \leq \frac{l_1 + l_2}{m}\right)$$

$$\quad\quad\quad\quad\quad w_{element4\_2}(x)$$

$$\quad\quad\quad\quad \text{else}$$

$$\quad\quad\quad\quad\quad \text{"not defined"}$$



$w_{second}(x)$

$$w_{11\_3}(x) := \left(3 \cdot \left(\frac{x}{\frac{\frac{l_1}{2}}{m}}\right)^2 - 2 \cdot \left(\frac{x}{\frac{\frac{l_1}{2}}{m}}\right)^3\right) \cdot \text{dn\_LinAlgEigenvectors}(A)_{1\,3} + \left(-\left(\frac{x}{\frac{\frac{l_1}{2}}{m}}\right)^2 + \left(\frac{x}{\frac{\frac{l_1}{2}}{m}}\right)^3\right) \cdot \frac{\frac{l_1}{2}}{m} \cdot \text{dn\_LinAlg}$$

$$w_{element1\_3}(x) := w_{11\_3}(x)$$

$$w_{12\_3}(x) := \left(1 - 3 \cdot \left(\frac{x}{\frac{l_1}{2}\,\mathrm{m}}\right)^2 + 2 \cdot \left(\frac{x}{\frac{l_1}{2}\,\mathrm{m}}\right)^3\right) \cdot \mathrm{dn\_LinAlgEigenvectors}(A)_{1\,3} + \left(\frac{x}{\frac{l_1}{2}\,\mathrm{m}} - 2 \cdot \left(\frac{x}{\frac{l_1}{2}\,\mathrm{m}}\right)^2 + \left(\frac{x}{\frac{l_1}{2}\,\mathrm{m}}\right)^3\right) \cdot \frac{\frac{l_1}{2}}{\mathrm{m}} \cdot$$

$$w_{element2\_3}(x) := w_{12\_3}\left(x - \frac{\frac{l_1}{2}}{\mathrm{m}}\right)$$

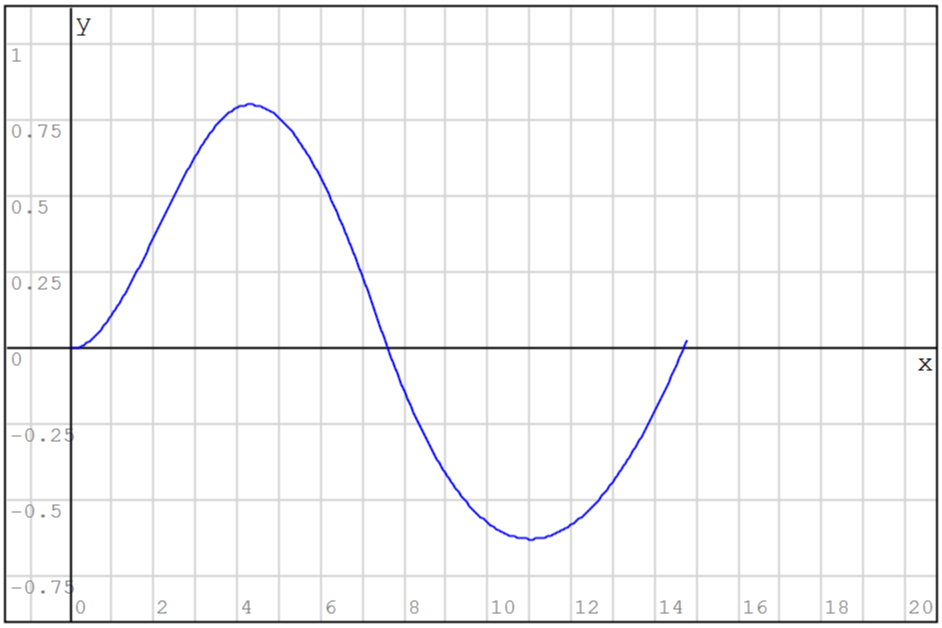
$$w_{21\_3}(x) := \left(1 - 3 \cdot \left(\frac{x}{\alpha \cdot l_2\,\mathrm{m}}\right)^2 + 2 \cdot \left(\frac{x}{\alpha \cdot l_2\,\mathrm{m}}\right)^3\right) \cdot \mathrm{dn\_LinAlgEigenvectors}(A)_{3\,3} + \left(\frac{x}{\alpha \cdot l_2\,\mathrm{m}} - 2 \cdot \left(\frac{x}{\alpha \cdot l_2\,\mathrm{m}}\right)^2 + \left(\frac{x}{\alpha \cdot l_2\,\mathrm{m}}\right.\right.$$

$$w_{element3\_3}(x) := w_{21\_3}\left(x - \frac{l_1}{\mathrm{m}}\right)$$

$$w_{22\_3}(x) := \left(1 - 3 \cdot \left(\frac{x}{(1-\alpha) \cdot l_2\,\mathrm{m}}\right)^2 + 2 \cdot \left(\frac{x}{(1-\alpha) \cdot l_2\,\mathrm{m}}\right)^3\right) \cdot \mathrm{dn\_LinAlgEigenvectors}(A)_{5\,3} + \left(\frac{x}{(1-\alpha) \cdot l_2\,\mathrm{m}} - 2 \cdot \left((1\right.\right.$$

$$w_{element4\_3}(x) := w_{22\_3}\left(x - \frac{l_1 + \alpha \cdot l_2}{\mathrm{m}}\right)$$

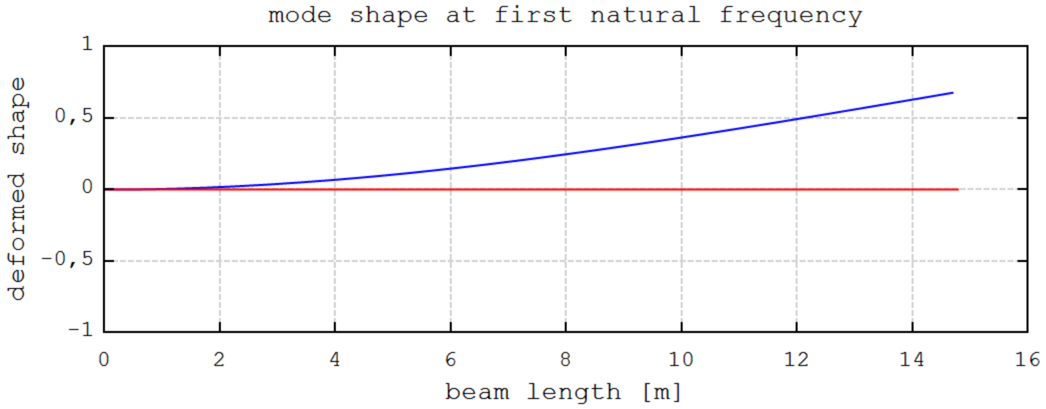
$$
\begin{aligned}
w_{third}(x) := &\ \mathrm{if}\ (x \ge 0) \wedge \left(x < \frac{\frac{l_1}{2}}{\mathrm{m}}\right)\\
&\quad w_{element1\_3}(x)\\
&\ \mathrm{else}\\
&\quad \mathrm{if}\ \left(x \ge \frac{\frac{l_1}{2}}{\mathrm{m}}\right) \wedge \left(x \le \frac{l_1}{\mathrm{m}}\right)\\
&\qquad w_{element2\_3}(x)\\
&\quad \mathrm{else}\\
&\qquad \mathrm{if}\ \left(x > \frac{l_1}{\mathrm{m}}\right) \wedge \left(x \le \frac{l_1 + \alpha \cdot l_2}{\mathrm{m}}\right)\\
&\qquad\quad w_{element3\_3}(x)\\
&\qquad \mathrm{else}\\
&\qquad\quad \mathrm{if}\ \left(x > \frac{l_1 + \alpha \cdot l_2}{\mathrm{m}}\right) \wedge \left(x \le \frac{l_1 + l_2}{\mathrm{m}}\right)\\
&\qquad\qquad w_{element4\_3}(x)\\
&\qquad\quad \mathrm{else}\\
&\qquad\qquad \text{"not defined"}
\end{aligned}
$$



$w_{third}(x)$

$$UNDEFORMD := \begin{bmatrix} l_1 + l_2 & 0 \\ 0 & 0 \end{bmatrix}$$

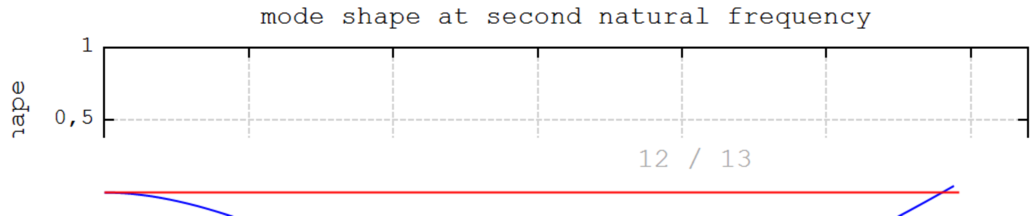
MODE SHAPES and NATURAL FREQUENCIES



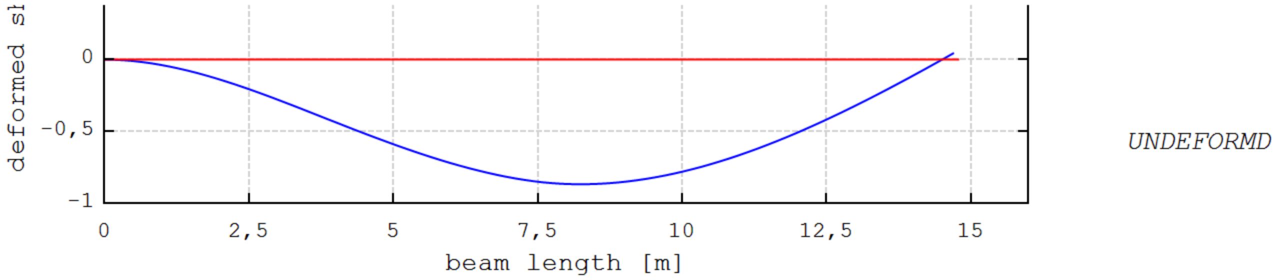
$f_1 = 0.669$ Hz

$$UNDEFORMD = \begin{bmatrix} 14.8\ \text{m} & 0 \\ 0 & 0 \end{bmatrix}$$

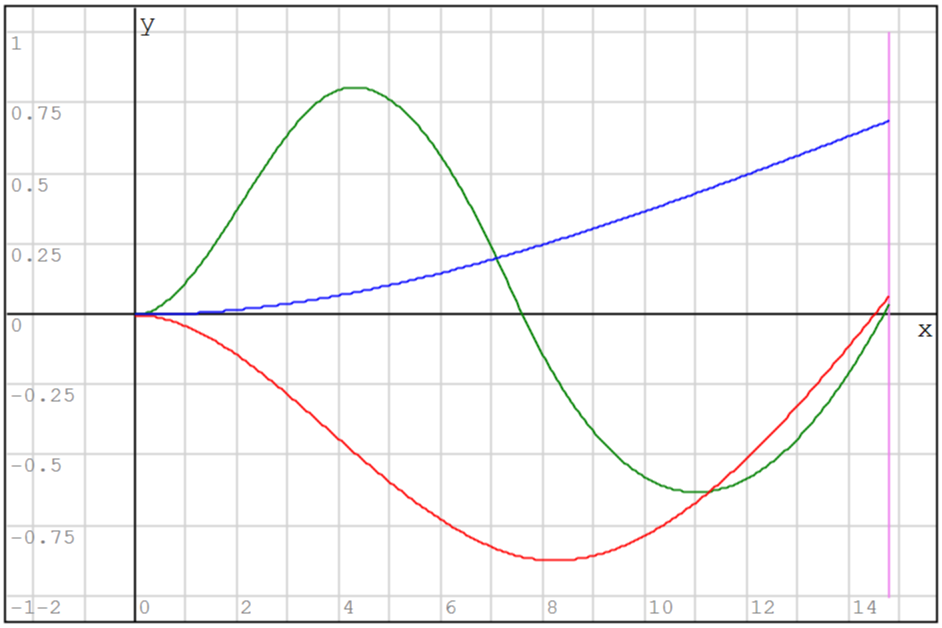
$\begin{cases} w_{first}(x) \\ UNDEFORMD \end{cases}$



$f_2 = 11.137$ Hz



$\begin{cases} w_{second}(x) \\ UNDEFORMD \end{cases}$



$$\begin{cases} w_{first}(x) \\ w_{second}(x) \\ w_{third}(x) \\ \begin{bmatrix} 14.8 & 1 \\ 014.8 & -1 \end{bmatrix} \end{cases}$$